II. **Solution by E. L. SHERWOOD, A. M., Superintendent of City Schools, West Point, Miss.**

Equation given $\rho=a\sin n\theta$. We may oberve that,

$$\rho=0,\ a,\ 0,\ -a,\ \text{etc., when}$$
$$\sin n\theta=0,\ 1,\ 0,\ -1,\ \text{etc., when}$$
$$n\theta=0,\ \tfrac{1}{2}\pi,\ \pi,\ \tfrac{3}{2}\pi,\ \text{etc., when}$$

$\theta=\{[c/n].\frac{1}{2}\pi\}$, where c is 0, 1, 2, 3, 4, etc., up to $4n$ ($4n$ being determined by $\theta=2\pi$).

The series of values will be as follows:

$$\theta=0.\frac{\pi}{2n},\quad \frac{\pi}{2n},\ 2.\frac{\pi}{2n},\ 3\frac{\pi}{2n}\ldots\ldots n.\frac{\pi}{2n},\ [n+1]\frac{\pi}{2n}\ldots\ldots 2n\frac{\pi}{2n},$$

$$[2n+1]\frac{\pi}{2n}\ \ldots\ ;$$

If n is *even*, $\rho=0,\ a,\ 0,\ -a\ldots\ldots 0\pm a\ldots\ldots 0,\ a$.
If n is *odd*, $\rho=0,\ a,\ 0,\ -a\ldots\ldots\pm a,\ 0\ldots\ldots 0,\ -a\ldots\ldots$

$$\left\{\begin{array}{cccc} 3n.\frac{\pi}{2n}, & [3n+1]\frac{\pi}{2n} & \ldots\ldots\ldots\ldots & 4n.\frac{\pi}{2n} \\ 0 & \pm a & \ldots\ldots\ldots\ldots & 0 \\ \pm a & 0 & \ldots\ldots\ldots\ldots & 0 \end{array}\right.$$

In each series are $4n$ terms (the first coincides with the last), and $\rho=a$ numerically in $2n$ of them. But when n is odd, the radius vector traces each loop twice for $\pi/2n$ and a is the same point as $\{[2n+1][\pi/2n]\}$ and $-a$.

$\therefore$ There are $2n$ loops when n is even, and n loops when n is odd.

$$\text{Area }=\tfrac{1}{2}\int\rho^2d\theta,\text{ where }\rho^2=a^2\sin^2n\theta,$$

$$=\tfrac{1}{2}a^2\int\sin^2nd\theta,$$

$$=\tfrac{1}{2}a^2\left[\tfrac{1}{2}\theta-\frac{\sin2n\theta}{4n}\right]_0^{\pi/2n}=\frac{\pi a^2}{8n}\text{ for }\tfrac{1}{2}\text{ loop,}$$

or $\pi a^2/4n$ for an entire loop.

$\therefore$ For $2n$ loops, area$=\pi a^2/2$; and for n loops, area$=\pi a^2/4$.

Also solved by *J. SCHEFFER* and *C. W. M. BLACK*.

62. Proposed by A. H. HOLMES, Brunswick, Maine.

A bucket is in the form of a frustum of a cone having its smaller end as a base. It is a inches in diameter at base and b inches in diameter at top, and its perpendicular height is c inches. It contains water the perpendicular height of which is $\frac{1}{2}c$ inches. What is the greatest height, from the plane on which the vessel rests, to which the surface of the water will rise when the bucket is overturned, no allowance being made for the thickness of the material of the bucket?

Partial Solution by C. W. M. BLACK, A. M., Professor of Mathematics in Wesleyan Academy, Wilbraham, Mass.

Let $AEFL$ be section of frustum. Complete the cone to the apex N. Let NM be axis, HK surface of water, PQ plane on which vessel rests, AB height of surface of water above PQ. Denote $\angle ONA$ by α, ON by l, HK by x, HM by y.

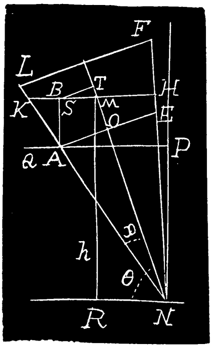

Fig. 1.

Then $EO=OA=\frac{1}{2}a$, $FL=b$. Denote angle which axis makes with PQ, $\angle ONR$ by θ, MR perpendicular to NR by h.

Now as vessel is tipped over, until H reaches E, volume of cone NHK is constant, and $=\frac{1}{3}\pi[\frac{1}{2}a+\frac{1}{4}b)^2[l+\frac{1}{2}c]$. Denote it by C.

Base HK is an ellipse, whose major axis is

$$x=h\cot[\theta-\alpha]-h\cot[\theta+\alpha] \quad\ldots\ldots\ldots\ldots(1).$$

Also $HM=y$, $=h\cot\theta+h\cot[\theta+\alpha]$ ....(2).

Let z=semi-minor axis, =ordinate in circular section through S, middle point of HK. Let r=radius of section.

Then $z=\sqrt{r^2-\overline{TS}^2}$, $=\sqrt{r^2-[(x/z)-y]^2\sin^2\theta}$ ....(3),

since $\angle TMS=\angle ONR$, $=\theta$. Also,

$$r=NT\tan\alpha,\ =\left[\frac{h}{\sin\theta}+\left(\frac{x}{2}-y\right)\cos\theta\right]\tan\alpha \quad\ldots\ldots(4).$$

Volume $NHK=[\frac{1}{3}\pi]h[x/2]$, $=C$ ....(5).

Let $\cot\theta=\beta$, $\cot\alpha=k$. (1) becomes, $x=\dfrac{2hk[\beta^2+1]}{k^2-\beta^2}$ ....(6);

(2) becomes, $y=\dfrac{h[\beta^2+1]}{k+\beta}$ ....(7), and $\dfrac{x}{2}-y=\dfrac{h\beta\,[\beta^2+1]}{k^2-\beta^2}$ ....(8).

From (4) by (8), $r=\dfrac{hk\sqrt{\beta^2+1}}{k^2-\beta^2}$ ....(9).

Substituting in (3),

$$z=\sqrt{\frac{h^2k^2[\beta^2+1]}{[k^2-\beta^2]^2}-\frac{h^2\beta^2[\beta^2+1]}{[k^2-\beta^2]^2}},\ =h\sqrt{\frac{\beta^2+1}{k^2-\beta^2}} \quad\ldots\ldots(10).$$

(5) becomes, $[\frac{1}{3}\pi]h\times\dfrac{hk[\beta^2+1]}{k^2-\beta^2}\times h\sqrt{\dfrac{\beta^2+1}{k^2-\beta^2}}=[\frac{1}{3}\pi]h^3k\left(\dfrac{\beta^2+1}{k^2-\beta^2}\right)^{\frac{3}{2}}=C$;

$$\text{whence } h=\sqrt[3]{\frac{3C}{\pi k}}\sqrt{\frac{k^2-\beta^2}{\beta^2+1}} \quad\ldots\ldots\ldots(11).$$

Now $AB = h - l\sin\theta + \frac{1}{2}a\cos\theta, = \sqrt[3]{\frac{3C}{\pi k}}\sqrt{\frac{k^2-\beta^2}{\beta^2+1}} - \frac{l}{\sqrt{\beta^2+1}} + \frac{a\beta}{2\sqrt{\beta^2+1}}$,

$$= \frac{\sqrt[3]{\frac{3C}{\pi k}}\sqrt{k^2-\beta^2} - l + \frac{a\beta}{2}}{\sqrt{\beta^2+1}} \quad \ldots\ldots\ \ldots\ldots\ldots\ldots(12),$$

in which β is the only variable. $dAB/d\beta =$

$$\frac{\sqrt{\beta^2+1}\left(-\sqrt[3]{\frac{3C}{\pi k}}\frac{\beta}{\sqrt{k^2-\beta^2}} + \frac{1}{2}a\right) - \left(\sqrt[3]{\frac{3C}{\pi k}}\sqrt{k^2-\beta^2} - l + \frac{a\beta}{2}\right)\frac{\beta}{\sqrt{\beta^2+1}}}{\beta^2+1} = 0$$

Equating to zero, and clearing of fractions,

$$(\beta^2+1)\left(-\sqrt[3]{\frac{3C}{\pi k}}\frac{\beta}{\sqrt{k^2-\beta^2}} + \frac{1}{2}a\right) - \left(\sqrt[3]{\frac{3C}{\pi k}}\sqrt{k^2-\beta^2} - l + \frac{a\beta}{2}\right)\beta = 0,$$

$$\text{or } \sqrt[3]{\frac{3C}{\pi k}}\left(\frac{\beta[\beta^2+1]}{\sqrt{k^2-\beta^2}} + \beta\sqrt{k^2-\beta^2}\right) = \frac{1}{2}a + l\beta.$$

Squaring and clearing, $[3C/\pi k]^{\frac{2}{3}}\,\beta^2[1+k^2]^2 = [\frac{1}{2}a + l\beta]^2[k^2-\beta^2]$. Whence,

$l^2\beta^4 + al\beta^3 + \{[3C/\pi k]^{\frac{2}{3}}\,[1+k^2]^2 + \frac{1}{4}a^2 - l^2k^2\}\beta^2 - alk^2\beta - \frac{1}{4}a^2k^2 = 0 \ldots\ldots\ \ldots(13)$.

Now $l = \frac{1}{2}a\cot\alpha, = \frac{1}{2}ak$; also, $k = \{c/\frac{1}{2}[b-a]\}, = \{2c/[b-a]\} \ldots\ldots\ldots(14)$.

$\therefore\ l = \dfrac{ac}{b-a}$. Also, $C = \pi\left(\dfrac{2a+b}{6}\right)^2[l + \frac{1}{3}c], = \pi\dfrac{[2a+b]^3c}{108[b-a]}$.

$\therefore\ 3C/\pi k = \{[2a+b]^3/216\} \ldots\ldots\ldots\ldots\ \ldots\ldots\ldots\ldots\ldots\ \ldots\ldots\ldots(16)$.

Substituting (14), (15), (16) in (13),

$$\frac{a^2c^2}{[b-a]}\beta^4 + \frac{a^2c}{b-a}\beta^3 + \left(\frac{[2a+b]^2\{[b-a]^2+4c^2\}}{36[b-a]^2}\right.$$

$$\left. + \frac{1}{4}a^2 - \frac{4a^2c^4}{[b-a]^4}\right)\beta^2 - \frac{4a^2c^3}{[b-a]^3}\beta - \frac{a^2c^2}{[b-a]^2} = 0 \quad \ldots\ \ldots(17).$$

By solving this for β we get the maximum values of AB, provided (Fig. 1) H does not pass E. In Fig. 2, representing this condition $\angle EAB = \theta$,

$\therefore\ AB = a\cos\theta$.

Accordingly (17) will produce critical values of θ, provided $\cos\theta$ is not $>$ AB/a, AB to be determined from (12).

It is evident that for any position of HK which cuts EA, the value of AB will be greater than that determined by the supposition made above. We must seek for maxima in this case by a different method.

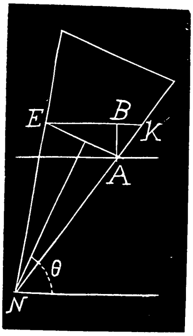

Fig. 2.

DKA (Fig. 3) represents section of volume of water.

$$\text{Volume} = [\pi c/9]\{[\tfrac{1}{4}a^2 + [\tfrac{1}{3}a + \tfrac{1}{6}b]^2 + \tfrac{1}{2}a[\tfrac{1}{3}a + \tfrac{1}{6}b]\},$$

$$= [\pi c/324][19a^2 + 7ab + b^2] \quad \ldots\ldots\ldots\ldots (18).$$

Equation (1)—(4) and (6)—(10) apply here as in Fig. 1.

Now volume ADK=cone NDK − cone NDA, $=\frac{1}{3}h\times$ [area elliptical segment DK] $-\frac{1}{3}l\times$ [area circular segment AD] $\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$ (19).

Let $AB=s$, $\angle DAC=\theta$, $\angle BKA=\theta-\alpha$.

$$DK = DB + BK, = s\tan\theta + s\cot[\theta-\alpha], = s\left(\frac{1}{\beta} + \frac{k\beta+1}{k-\beta}\right), = \left(\frac{sk[1+\beta^2]}{\beta[k-\beta]}\right) \quad \ldots (20).$$

$$\text{Area segment } DK = \frac{\pi zx}{2} - \frac{zx}{2}\left(\cos^{-1}\frac{[DK-\frac{1}{2}x]}{\frac{1}{2}x} - \frac{DK-\frac{1}{2}x}{\frac{1}{2}x}\sqrt{1-\left(\frac{DK-\frac{1}{2}x}{\frac{1}{2}x}\right)^2}\right).$$

Substitute from (6), (10), and (20):

$$\text{Area } DK = \frac{h^2k[\beta^2+1]^{\frac{3}{2}}}{[k^2-\beta^2]^{\frac{3}{2}}}\left[\pi - \cos^{-1}\left(\frac{s[k+\beta]}{h\beta} - 1\right)\right.$$

$$\left. + \left(\frac{s[k+\beta]}{h\beta} - 1\right)\sqrt{\frac{s[k+\beta]}{h\beta}\left(2 - \frac{s[k+\beta]}{h\beta}\right)}\ \right] \quad \ldots (21).$$

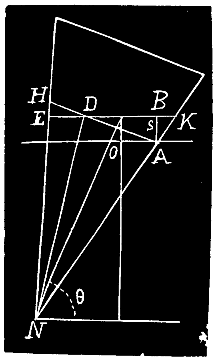

Fig. 3.

Now $s = h - l\sin\theta + \frac{1}{2}a\cos\theta$.

$$\therefore\ h = s + l\sin\theta - \tfrac{1}{2}a\cos\theta, = s + \frac{2l - a\beta}{2\sqrt{\beta^2+1}} \quad \ldots\ldots (22).$$

$$AD = s\sec\theta, = \frac{s\sqrt{\beta^2+1}}{\beta}.$$

$$\text{Area segment } AD = \tfrac{1}{4}a^2\left[\pi - \cos^{-1}\left(\frac{2s\sqrt{\beta^2+1}}{a\beta} - 1\right)\right.$$

$$\left. + 2\left(\frac{2s\sqrt{\beta^2+1}}{a\beta} - 1\right)\sqrt{\frac{s\sqrt{\beta^2+1}}{a\beta}\left(1 + \frac{s\sqrt{\beta^2+1}}{a\beta}\right)}\right] \quad \ldots\ldots (23).$$

Substitute from (18), (21), (22), and (23) in (19),

$$\frac{k[s\sqrt{\beta^2+1}+l-\frac{1}{2}a\beta]^3}{3[k^2-\beta^2]^{\frac{3}{2}}}\left\{\pi-\cos^{-1}\left[\frac{s[k+\beta]}{\beta\{s+[(2l-a\beta)/(2\sqrt{1+\beta^2})]\}}-1\right]\right.$$

$$+\left[\frac{s[+\beta]}{\beta\{s+[(2l-a\beta)/(2\sqrt{\beta^2+1})]\}}-1\right]\times$$

$$\left.\sqrt{\frac{s[k+\beta]}{\beta\{s+[(2l-a\beta)/(2\sqrt{\beta^2+1})]\}}\left[2-\frac{s[k+\beta]}{\beta\{s+[(2l-a\beta)/(2\sqrt{\beta^2+1})]\}}\right]}\right\}$$

$$-\frac{a^3k}{24}\left\{\pi-\cos^{-1}\left[\frac{2s\sqrt{\beta^2+1}}{a\beta}-1\right]\right.$$

$$\left.+2\left[\frac{2s\sqrt{\beta^2+1}}{a\beta}-1\right]\sqrt{\frac{s\sqrt{\beta^2+1}}{a\beta}\left[1-\frac{s\sqrt{\beta^2+1}}{a\beta}\right]}\right\}=\tfrac{1}{324}\pi c[19a^2+7ab+b^2],$$

which equation contains only s, β, and constants. However, the chance of solving it after differentiation seems extremely slight.

PROBLEMS FOR SOLUTION.

MISCELLANEOUS.

56. **Proposed by S. HART WRIGHT, A. M., M. D., Ph. D., Penn Yan, N. Y.**

In latitude 40° N.$=\lambda$, when the moon's declination is 5° 23′ N.$=\delta$, and the sun's declination 9° 52′ S.$=-\delta'$, how long after sunset will the cusps of the moon's crescent set synchronously, the moon having recently passed its conjunction with the sun ?

57. **Proposed by GEORGE LILLEY, Ph. D., LL. D., Professor of Mathematics in the Oregon State University, Eugene, Oregon.**

A particle is placed very near the center of a circle, round the circumference of which n equal repulsive forces are symmetrically arranged ; each force varies inversely as the mth power of its distance from the particle. Show that the resultant force is approximately $\frac{m_1 n(m-1)}{2r^{m+1}}\times CP$, and tends to the center of the circle, where m_1 is the mass of the particle, CP its distance from the center of the circle, and r the radius of the circle.

EDITORIALS.

The credit of preparing the index for this volume is due Editor Colow.

Dr. Artemas Martin, of the U. S. Coast and Geodetic Survey, has been